Unit 4: Theory of Radiation

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Abstract

Theory of Radiation: Blackbody radiation, Spectral distribution, Concept of Energy Density, Derivation of Planck's law, Deduction of Wiens distribution law, Rayleigh-Jeans Law, Stefan Boltzmann Law and Wiens displacement law from Plancks law.

1 Blackbody radiation

A black-body is an idealised object which absorbs and emits all radiation frequencies. Near thermodynamic equilibrium, the emitted radiation is closely described by Planck's law and because of its dependence on temperature, Planck radiation is said to be thermal radiation, such that the higher the temperature of a body the more radiation it emits at every wavelength.

2 Rayleigh Jeans Formula

Rayleigh Jeans developed a theory for the spectral distribution of blackbody radiation. Consider a cavity of length l that behaves as a blackbody, such that radiation in the form of standing waves are trapped in the cavity. The wavelengths of these radiations are given by

$$n\lambda/2 = l$$

where n=1,2,3.... The corresponding frequencies (overtones) are

$$\nu = c/(\lambda) = \frac{cn}{2l}$$

where n=1,2,3.... Here c is the velocity of light

The number of overtones may be referred to as the several degrees of freedom with which the system is capable of vibrating. Every allowed frequency is called a mode of vibration.

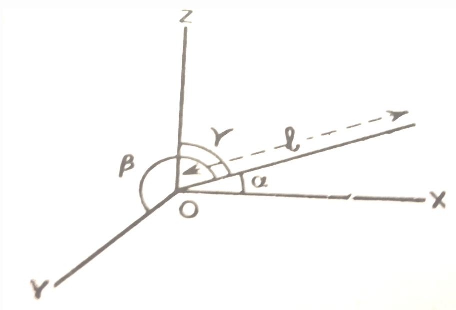

Figure 1: Fig1

If the three edges of the cube from the three axis in space, the number of loops in each direction are n_x, n_y, n_z

$$n_x = \frac{2l}{\lambda}$$

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$$l = n_x(\lambda/2) = n_y(\lambda/2) = n_z(\lambda/2)$$

As the radiation are diffuse, the waves are inclined at angles α, β, γ to the x,y,z axis.

$$lcos\alpha = n_x(\lambda/2)$$

$$lcos\beta = n_y(\lambda/2)$$

$$lcos\gamma = n_z(\lambda/2)$$

Also

$$cos^2\alpha + cos^2\beta + cos^2\gamma = 1$$

$$n_x^2 + n_y^2 + n_z^2 = (\frac{2l}{\lambda})^2$$

If

$$n_x^2 + n_y^2 + n_z^2 = r^2, r^2 = (\frac{2l}{\lambda})^2$$

,

$$r = \frac{2l}{\lambda}$$

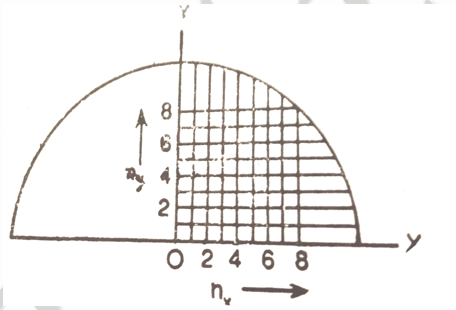

Figure 2: Fig 2

The above equation is an ellipsoid with n_x, n_y, n_z directions as coordinate axis. The total n_i=number of modes of vibrations are the possible sets of n_x, n_y, n_z. In the two dimensional case,

$$n_x^2 + n_y^2 = (\frac{2l}{\lambda})^2$$

This equation represents a circle with n_x, n_y as in the figure 2, the number of squares is equal to the area of the quadrant $1/4\pi r^2$

$$(1/4)\pi(\frac{2l}{\lambda})^2 = \frac{\pi l^2}{\lambda^2}$$

We extend this to the 3D case, then the volume of the spherical shell will be

$$f = 1/8(4/3\pi r^3) = 1/8(4/3\pi(\frac{2l}{\lambda})^3 = \frac{4\pi l^3}{3\lambda^3}$$

The number of possible modes of vibration

$$df = \frac{4\pi l^3}{3}\frac{d}{d\lambda}(1/\lambda^3) = \frac{4\pi l^3}{3}(-\frac{3}{\lambda^4}d\lambda)$$

Neglecting negative values of $d\lambda$,

$$df = \frac{4\pi V}{\lambda^4}d\lambda$$

as $l^3 = V$.

No of vibrations per unit volume,

$$= \frac{4\pi}{\lambda^4}d\lambda$$

With two polarizations per mode,

$$= \frac{8\pi}{\lambda^4}d\lambda$$

Rayleigh and Jeans assumed that the law of equipartition of energy holds good in case of radiation also. According to this law, the average energy per mode of vibration is kT where k is Boltzman's constant and T is absolute temperature.

Energy density within wavelength λ and $\lambda + d\lambda$ is $E_\lambda d\lambda$= Total no of modes of vibration X Average energy per mode

$$= \frac{8\pi}{\lambda^4} d\lambda \times kT$$
$$= \frac{8\pi kT}{\lambda^4} d\lambda$$

This is Rayleigh-Jeans Formula.

Rayleigh-Jeans law accurately predicts experimental results at radiative frequencies below 10^5 GHz, but diverges with observations when frequencies reach the ultraviolet region of the electromagnetic spectrum (small λ). This was called the 'ultraviolet catastrophe', as the energy would be very large at the ultraviolet region according to theory, but was much smaller according to observations.

Plank's Law finally resolved this problem.

3 Plank's Law

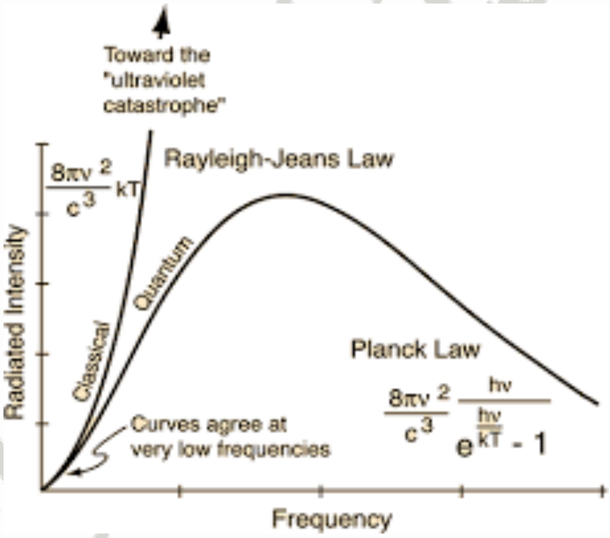

Figure 3: Plancks radiation law

Planck's law describes the spectral density of electromagnetic radiation emitted by a black body in thermal equilibrium at a given temperature T, when there is no net flow of matter or energy between the body and its environment.

At the end of the 19th century, physicists were unable to explain why the observed spectrum of black body radiation, which by then had been accurately measured, diverged significantly at higher frequencies from that predicted by existing theories. In 1900, Max Planck derived a formula for the observed spectrum by assuming that a hypothetical electrically charged oscillator in a cavity that contained black-body radiation could only change its energy in a minimal increment, E, that was proportional to the frequency of its associated electromagnetic wave. This resolved the problem of the ultraviolet catastrophe predicted by classical physics.

It was a pioneering insight of modern physics and is of fundamental importance to quantum theory.

Plank made the following three assumptions: 1. A black body radiator contains simple harmonic oscillators of possible frequencies

2. The oscillators cannot absorb or emit energy continuously.

3. Energy can be absorbed or emitted only in discrete packets of $nh\nu$ where n is an integer 1,2,3....

Let N be the number of Plank Oscillators and E be their total energy. Then the average energy per oscillator

$$\epsilon = \frac{E}{N}$$

Let there be $N_0, N_1, N_2 ......N_r .....$ oscillators with energy $0, \epsilon, 2\epsilon, 3\epsilon ....r\epsilon$ respectively. Now

$$N = N_0 + N_1 + N_2 + ... + N_r + ...$$

$$E = 0 + \epsilon N_1 + 2\epsilon N_2 + ... + 3\epsilon N_3 + r\epsilon N_r + ...$$

From Maxwells distribution formula

$$N_r = N_0 e^{-\frac{\epsilon_r}{kT}}$$

Substituting

$$N = N_0 + N_0 e^{-\frac{\epsilon}{kT}} + N_0 e^{-\frac{2\epsilon}{kT}} + ... + N_0 e^{-\frac{r\epsilon}{kT}} + ...$$

$$= N_0 \left[1 + e^{-\frac{\epsilon}{kT}} + e^{-\frac{2\epsilon}{kT}} + ... + e^{-\frac{r\epsilon}{kT}} + ...\right]$$

$$= \frac{N_0}{\left[1 - e^{\frac{\epsilon}{kT}}\right]}$$

Since

$$1 + x + x^2 + x^3 + ... = \frac{1}{(1-x)}$$

Substituting

$$E = (N_0 \times 0) + \epsilon N_0 e^{-\frac{\epsilon}{kT}} + 2\epsilon N_0 e^{-\frac{2\epsilon}{kT}} + ... + r\epsilon N_0 e^{-\frac{r\epsilon}{kT}} + ...$$

$$= N_0 \epsilon e^{-\frac{\epsilon}{kT}} \left[1 + 2e^{-\frac{\epsilon}{kT}} + 3e^{-\frac{2\epsilon}{kT}} + ... + re^{-\frac{(r-1)\epsilon}{kT}} + ...\right]$$

$$= N_0 \epsilon e^{-\frac{\epsilon}{kT}} \left[\frac{1}{(1 - e^{-\frac{\epsilon}{kT}})^2}\right]$$

Since

$$1 + 2x + 3x^2 + x^3 + ... = \frac{1}{(1-x)^2}$$

Now the average energy per oscillator

$$\epsilon = \frac{E}{N} = \frac{N_0 \epsilon e^{-\frac{\epsilon}{kT}} \left[\frac{1}{(1-e^{-\frac{\epsilon}{kT}})^2}\right]}{\frac{N_0}{\left[1-e^{\frac{\epsilon}{kT}}\right]}}$$

$$= \frac{\epsilon e^{-\frac{\epsilon}{kT}}}{1 - e^{\frac{\epsilon}{kT}}}$$

$$\frac{\epsilon}{e^{\frac{\epsilon}{kT}} - 1}$$

Now

$$\epsilon = h\nu$$

$$= \frac{h\nu}{e^{\frac{h\nu}{kT}} - 1}$$

From the earlier derivation, we know the number of oscillators per unit volume is

$$= \frac{8\pi}{\lambda^4} d\lambda = \frac{8\pi\nu^2}{c^3} d\nu$$

Since

$$\lambda = \frac{c}{\nu}, d\lambda = -\frac{c}{\nu^2}$$

Therefore

$$E_\nu d\nu = \frac{8\pi\nu^2}{c^3} d\nu \frac{h\nu}{e^{\frac{h\nu}{kT}} - 1}$$

In terms of wavelength

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1} d\lambda$$

This is Plancks radiation law

4 Derivation of Rayleigh-Jeans Law and Wiens Law from Planks Law

4.1 Wiens Law

For short wavelengths, when λ is very small,

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}d\lambda$$

then

$$e^{\frac{hc}{\lambda kT}}-1 = e^{\frac{hc}{\lambda kT}}$$

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}}d\lambda$$

Putting

$$8\pi hc = A, \frac{hc}{k} = B$$

$$E_\lambda d\lambda = \frac{A}{\lambda^5}e^{-\frac{B}{\lambda T}}d\lambda$$

This is Wien's Formula

4.2 Rayleigh-Jeans Law

For long wavelengths, when λ is very large,

$$e^{\frac{hc}{\lambda kT}} = 1 + \frac{hc}{\lambda kT} + \left(\frac{hc}{\lambda kT}\right)^2 + ...$$

$$= 1 + \frac{hc}{\lambda kT}$$

Now

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}d\lambda$$

$$= \frac{8\pi hc}{\lambda^5}\frac{1}{(1+\frac{hc}{\lambda kT})-1}d\lambda$$

$$= \frac{8\pi hc}{\lambda^5}\frac{\lambda kT}{hc}d\lambda$$

$$= \frac{8\pi kT}{\lambda^4}d\lambda$$

4.3 Wiens Displacement Law

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}d\lambda$$

Putting

$$8\pi hc = A$$

$$E_\lambda = A\lambda^{-5}\frac{1}{e^{\frac{hc}{\lambda kT}}-1}$$

To find the wavelength at which we have maximum radiation, we take the derivative and equate it to zero

$$A\left[\lambda^{-5}(-1)\left(\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\right)^{-2}e^{\frac{hc}{\lambda kT}}\frac{-hc}{\lambda^2 kT} - 5\lambda^{-6}\left(\frac{1}{e^{\frac{hc}{\lambda kT}}-1}\right)^{-1}\right] = 0$$

$$\left(e^{\frac{hc}{\lambda kT}}-1\right)^{-1}e^{\frac{hc}{\lambda kT}}\frac{hc}{\lambda^2 kT} = \frac{5}{\lambda}$$

$$\left(\frac{hc}{\lambda kT}\right)\frac{e^{\frac{hc}{\lambda kT}}}{\left(e^{\frac{hc}{\lambda kT}}-1\right)} = 5$$

Putting

$$\frac{hc}{\lambda kT} = x$$

$$\frac{xe^x}{e^x - 1} = 5$$

$$\frac{x}{5} + e^x = 1$$

the two roots are x=0 and x=4.965 using

$$\frac{hc}{\lambda kT} = 4.965$$

$$\lambda_{max}T = \frac{hc}{4.965k} = b$$

4.4 Stefans Law

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}} - 1}d\lambda$$

The total energy radiated will be the integral of the above equation.

$$E = \int_0^\infty E_\nu = \frac{2\pi h}{c^2}\int_0^\infty \frac{\nu^3 d\nu}{e^{\frac{h\nu}{kT}} - 1}$$

Let

$$\frac{h\nu}{kT} = x, \nu = \frac{kTx}{h}, d\nu = \frac{kTdx}{h}$$

Substituting,

$$E = \frac{2\pi(kT)^4}{h^3c^2}\int_0^\infty \frac{x^3 dx}{e^x - 1}$$

$$E = \frac{2\pi(kT)^4}{h^3c^2} \times \frac{\pi^4}{15}$$

Since

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$$

$$= \frac{2\pi^5 k^4}{15h^3c^2}T^4$$

$$E = \sigma T^4$$

where $\sigma = \frac{2\pi^5 k^4}{15h^3c^2}$