

23-8-19

## PRINCIPLE OF SUPERPOSITION OF WAVES.

The wave motion has one important property that is when two or more than two waves in the same medium pass through one another they pass in such a way as if the other wave is not present.

Wave motion کی ایک اہم خاصیت یہ ہوتی ہے کہ جب دو یا دو سے زیادہ لہریں ایک ہی (same) ذریعہ (medium) سے گزرتی ہو اس طرح کہ کوئی دوسری لہر موجود نہیں ہے۔

Statement :- (بیان)

اگر  $y_1, y_2, y_3$  تین displacement vectors 1, 2, 3 لہریں دو

تب resultant متعلقہ  $y$  اس طرح دی جاسکتی ہے  

$$y = y_1 + y_2 + y_3$$

Superposition of waves کے لیے اہم Cases مندرجہ ذیل

(1) دو لہریں جیسا کہ same مقدار میں اور same سمت میں حرکت کرتی ہیں۔  
 (Interference of waves).

(2) دو لہریں جس کا مقدار slightly مختلف ہو اور جو ایک ہی سمت میں حرکت کرتی ہیں۔  
 (Beats).

(3) دو لہریں جس کا مقدار same ہو اور مختلف سمت میں حرکت کرتی ہیں۔  
 ✓ (Stationary waves).

## Linear Superposition of two waves of same frequency

فرض کیجئے کہ ایک S.H.M. کا ایک case جس کا دور (Same Period) ہے۔  
 (Same) مقدار میں  $x$  کے same سمت میں  $a_1$  اور  $a_2$  amplitude کے ساتھ حرکت کرتے ہیں۔ اور جس کا phase difference  $\phi$  ہے۔



$$a_1^2 + a_2^2 (\sin^2 \phi + \cos^2 \phi) + 2a_1 a_2 \cos \phi = A^2$$

$$a_1^2 + a_2^2 + 2a_1 a_2 \cos \phi = A^2$$

$$A = (a_1^2 + a_2^2 + 2a_1 a_2 \cos \phi)^{1/2}$$

اسی کی بھی نقطہ پر کا resultant amplitude اس نقطہ پر ملنے والے  
2 لہروں کے phase فرق پر منحصر ہوتی ہیں۔

$\theta$  کی قدر مساوات 3 اور 4 کو تقسیم کر کے حاصل کی جاسکتی ہے

$$\frac{A \sin \theta}{A \cos \theta} = \frac{a_2 \sin \phi}{a_1 + a_2 \cos \phi}$$

$$(\because \sin \theta / \cos \theta = \tan \theta)$$

$$\tan \theta = \frac{a_2 \sin \phi}{a_1 + a_2 \cos \phi}$$

$$\theta = \tan^{-1} \frac{a_2 \sin \phi}{a_1 + a_2 \cos \phi} \quad \text{--- (6)}$$

### Special Cases:-

(1) Superposition of two waves in the same phase.  
i.e.  $\phi = 0$ .

اس Case میں resultant amplitude نکال دیا جائے گا۔

$$A = (a_1^2 + a_2^2 + 2a_1 a_2 \cos(0))^{1/2}$$

$$= (a_1^2 + a_2^2 + 2a_1 a_2)^{1/2}$$

$$(\because \cos 0 = 1)$$

$$= (a_1 + a_2)^2 / 2$$

$$= a_1 + a_2$$

(2) Superposition of two waves in the opposite phase  
i.e.  $\phi = \pi$

$$A = (a_1^2 + a_2^2 + 2a_1 a_2 \cos \pi)^{1/2}$$

$$(\because \cos \pi = -1)$$

$$A = (a_1^2 + a_2^2 - 2a_1 a_2)^{1/2}$$

$$A = (a_1 - a_2)$$



$$a_1^2 + a_2^2 (\sin^2 \phi + \cos^2 \phi) + 2a_1 a_2 \cos \phi = A^2$$

$$a_1^2 + a_2^2 + 2a_1 a_2 \cos \phi = A^2$$

$$A = (a_1^2 + a_2^2 + 2a_1 a_2 \cos \phi)^{1/2}$$

اسی کی بھی نقطہ پر کا resultant amplitude اس نقطہ پر ملنے والے  
2 لہروں کے phase فرق پر منحصر ہوتی ہیں۔

$\theta$  کی قدر مساوات 3 اور 4 کو تقسیم کر کے حاصل کی جاسکتی ہے

$$\frac{A \sin \theta}{A \cos \theta} = \frac{a_2 \sin \phi}{a_1 + a_2 \cos \phi}$$

$$(\because \sin \theta / \cos \theta = \tan \theta)$$

$$\tan \theta = \frac{a_2 \sin \phi}{a_1 + a_2 \cos \phi}$$

$$\theta = \tan^{-1} \frac{a_2 \sin \phi}{a_1 + a_2 \cos \phi} \quad \text{--- (6)}$$

### Special Cases:-

(1) Superposition of two waves in the same phase.  
i.e.  $\phi = 0$ .

اس Case میں resultant amplitude نکال دیا جائے گا۔

$$A = (a_1^2 + a_2^2 + 2a_1 a_2 \cos(0))^{1/2}$$

$$= (a_1^2 + a_2^2 + 2a_1 a_2)^{1/2}$$

$$(\because \cos 0 = 1)$$

$$= (a_1 + a_2)^2 / 2$$

$$A = a_1 + a_2$$

(2) Superposition of two waves in the opposite phase  
i.e.  $\phi = \pi$

$$A = (a_1^2 + a_2^2 + 2a_1 a_2 \cos \pi)^{1/2}$$

$$(\because \cos \pi = -1)$$

$$A = (a_1^2 + a_2^2 - 2a_1 a_2)^{1/2}$$

$$A = |a_1 - a_2|$$



اور اگر  $a_1$  amplitude ہے sound waves  
اور  $a_2$  amplitude ہے sound waves

But, if they arrive in <sup>opp</sup> ~~same~~ phase, the resultant amplitude will be  $(a_1 - a_2)$  and hence a feeble sound.

✓ BEATS

from two tuning fork | Slightly different (جب دو لہریں ہلکا سا مختلف ہوں)

تب اس کے resultant amplitude کی وجہ اور (maximum & minimum) کی وجہ سے

# Mathematical Analysis

Consider the case of two waves having same amplitude  $a$  with slightly different frequency  $n_1$  and  $n_2$  travelling simultaneously in a medium.

If  $y_1$  and  $y_2$  be the displacement of these waves at any instant  $t$ , then

$$y_1 = a \sin 2\pi n_1 t$$

$$y_2 = a \sin 2\pi n_2 t$$



$$\cos 2\pi \left( \frac{n_1 - n_2}{2} \right) t = \pm 1$$

$$2\pi \left( \frac{n_1 - n_2}{2} \right) t = K\pi$$

$K = 0, 1, 2$  جہاں

$$t = \frac{K}{(n_1 - n_2)}$$

جہاں  $K = 0, t = 0$  First maxima

جب  $K = 1$

$$t_1 = \frac{1}{(n_1 - n_2)}$$

2<sup>nd</sup> maxima

جب  $K = 2$

$$t_2 = \frac{2}{(n_1 - n_2)}$$

3<sup>rd</sup> maxima

جب  $K = n$

$$t = \frac{n}{(n_1 - n_2)}$$

$n^{\text{th}}$  maxima

Thus the time interval b/w two successive maxima

$$= \frac{1}{(n_1 - n_2)} \text{ seconds.}$$

$\therefore$  Frequency of maxima =  $(n_1 - n_2)$   
Similarly, the amplitude is maximum, when

$$\cos 2\pi \left( \frac{n_1 - n_2}{2} \right) t = 0$$

$$t = \frac{(2K+1)\pi}{2\left(\frac{n_1 - n_2}{2}\right)}$$

$$2\pi \left( \frac{n_1 - n_2}{2} \right) t = (2K+1) \frac{\pi}{2}$$

$$t = \frac{2K+1}{2(n_1 - n_2)}$$

جب  $K = 0$

$$t = \frac{1}{2(n_1 - n_2)}$$



$$\cos 2\pi \left( \frac{n_1 - n_2}{2} \right) t = \pm 1$$

$$2\pi \left( \frac{n_1 - n_2}{2} \right) t = K\pi$$

$K = 0, 1, 2$  جہاں

$$t = \frac{K}{(n_1 - n_2)}$$

جہاں  $K = 0, t = 0$  First maxima

جب  $K = 1$

$$t_1 = \frac{1}{(n_1 - n_2)}$$

2<sup>nd</sup> maxima

جب  $K = 2$

$$t_2 = \frac{2}{(n_1 - n_2)}$$

3<sup>rd</sup> maxima

جب  $K = n$

$$t = \frac{n}{(n_1 - n_2)}$$

$n^{\text{th}}$  maxima

Thus the time interval b/w two successive maxima

$$= \frac{1}{(n_1 - n_2)} \text{ seconds.}$$

$\therefore$  Frequency of maxima =  $(n_1 - n_2)$

Similarly, the amplitude is maximum, when

$$\cos 2\pi \left( \frac{n_1 - n_2}{2} \right) t = 0$$

$$t = \frac{(2K+1)\pi}{2\left(\frac{n_1 - n_2}{2}\right)}$$

$$2\pi \left( \frac{n_1 - n_2}{2} \right) t = (2K+1) \frac{\pi}{2}$$

$$t = \frac{2K+1}{2(n_1 - n_2)}$$

جب  $K = 0$

$$t = \frac{1}{2(n_1 - n_2)}$$



$$t = \frac{2+1}{2(n_1 - n_2)}$$

$$= \frac{3}{2(n_1 - n_2)}$$

$$K = 2$$

$$t = \frac{2(2)+1}{2(n_1 - n_2)}$$

$$= \frac{5}{2(n_1 - n_2)}$$

$$K = n$$

$$t_n = \frac{n+1}{2(n_1 - n_2)}$$

The interval b/w two successive minima.

$$= \frac{1}{(n_1 - n_2)} \text{ sec.}$$

$$\therefore \text{Frequency of minima} = (n_1 - n_2)$$

Hence  $(n_1 - n_2)$  beats will be heard in 1 second.



$$\frac{x}{a} - \frac{y}{b} \cos \phi = \sqrt{1 - \frac{y^2}{b^2}} \sin \phi$$

Squaring on both sides

$$\left( \frac{x}{a} - \frac{y}{b} \cos \phi \right)^2 = \left( \sqrt{1 - \frac{y^2}{b^2}} \right)^2 \sin^2 \phi$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \cos^2 \phi - 2 \frac{x}{a} \cdot \frac{y}{b} \cos \phi = \left( 1 - \frac{y^2}{b^2} \right) \sin^2 \phi$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} (\cos^2 \phi + \sin^2 \phi) - 2 \frac{x}{a} \cdot \frac{y}{b} \cos \phi = \sin^2 \phi$$

$$(\because \cos^2 \phi + \sin^2 \phi = 1)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2 \frac{x}{a} \cdot \frac{y}{b} \cos \phi = \sin^2 \phi \quad - (3)$$

مسلمات (3) oblique ellipse کا خارج کرتا ہے۔ (Particle)  $\phi = 0$  resultant path

Here we consider the following important cases.

(i) جب (two variation are in phase)  $\phi = 0$  اس Case میں

$$\sin \phi = 0 \quad \cos \phi = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2 \frac{x}{a} \cdot \frac{y}{b} = 0 \quad \text{اب مسلمات (3)}$$

$$\left( \frac{x}{a} - \frac{y}{b} \right)^2 = 0$$

or

$$\pm \left( \frac{x}{a} - \frac{y}{b} \right) = 0$$

$$\pm y = \pm \frac{b}{a} x$$

This represent two coincident straight line passing through



$$\frac{x}{a} - \frac{y}{b} \cos \phi = \sqrt{1 - \frac{y^2}{b^2}} \sin \phi$$

Squaring on both sides

$$\left( \frac{x}{a} - \frac{y}{b} \cos \phi \right)^2 = \left( \sqrt{1 - \frac{y^2}{b^2}} \right)^2 \sin^2 \phi$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \cos^2 \phi - 2 \frac{x}{a} \cdot \frac{y}{b} \cos \phi = \left( 1 - \frac{y^2}{b^2} \right) \sin^2 \phi$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} (\cos^2 \phi + \sin^2 \phi) - 2 \frac{x}{a} \cdot \frac{y}{b} \cos \phi = \sin^2 \phi$$

$$(\because \cos^2 \phi + \sin^2 \phi = 1)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2 \frac{x}{a} \cdot \frac{y}{b} \cos \phi = \sin^2 \phi \quad - (3)$$

مسلمات (3) oblique ellipse کا خارج کرتا ہے۔ (Particle)  $\phi = 0$  resultant path

Here we consider the following important cases.

(i) جب (two variation are in phase)  $\phi = 0$  اس Case میں

$$\sin \phi = 0 \quad \cos \phi = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2 \frac{x}{a} \cdot \frac{y}{b} = 0 \quad \text{اب مسلمات (3)}$$

$$\left( \frac{x}{a} - \frac{y}{b} \right)^2 = 0$$

or

$$\pm \left( \frac{x}{a} - \frac{y}{b} \right) = 0$$

$$\pm y = \pm \frac{b}{a} x$$

This represent two coincident straight line passing through



so the resultant path of the particle is a circle of a radius  $a$  as shown in Fig 9 (d)

$$\phi = 3\pi/4 \quad \text{or} \quad (iv)$$

$$\sin \phi = \frac{1}{\sqrt{2}} \quad \& \quad \cos \phi = -\frac{1}{\sqrt{2}}$$

Then Eq (3) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} \left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$$

This represents an oblique ellipse as shown in Fig. 9(e)

$$\phi = \pi \quad \text{or} \quad (v)$$

we have

$$\sin \phi = 0 \quad \text{and} \quad \cos \phi = -1$$

Now Eq (3) reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2xy}{ab} = 0$$

$$\left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0$$

$$\pm \left(\frac{x}{a} + \frac{y}{b}\right) = 0$$

$$\pm y = \mp \frac{b}{a} x$$

This again represents a pair of coincident straight lines passing through the origin & inclined to  $x$  axis at an angle  $\theta$ , given by

$$\theta = \tan^{-1} \left(-\frac{b}{a}\right)$$



so the resultant path of the particle is a circle of a radius  $a$  as shown in Fig 9 (d)

$$\phi = 3\pi/4 \quad \text{or} \quad (iv)$$

$$\sin \phi = \frac{1}{\sqrt{2}} \quad \& \quad \cos \phi = -\frac{1}{\sqrt{2}}$$

Then Eq (3) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} \left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$$

This represents an oblique ellipse as shown in Fig. 9(e)

$$\phi = \pi \quad \text{or} \quad (v)$$

we have

$$\sin \phi = 0 \quad \text{and} \quad \cos \phi = -1$$

Now Eq (3) reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2xy}{ab} = 0$$

$$\left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0$$

$$\pm \left(\frac{x}{a} + \frac{y}{b}\right) = 0$$

$$\pm y = \mp \frac{b}{a} x$$

This again represents a pair of coincident straight lines passing through the origin & inclined to  $x$  axis at an angle  $\theta$ , given by

$$\theta = \tan^{-1} \left(-\frac{b}{a}\right)$$



$$\frac{x}{a} = 2 \sin \omega t \cos \omega t \cdot \cos \phi + (1 - 2 \sin^2 \omega t) \sin \phi$$

$\uparrow$   $\cos \omega t$   $\uparrow$   $\sin \omega t$

$$\frac{x}{a} = 2 \frac{y}{b} \sqrt{1 - \frac{y^2}{b^2}} \cos \phi + \left(1 - \frac{2y^2}{b^2}\right) \sin \phi$$

$$\frac{x}{a} - \left(1 - \frac{2y^2}{b^2}\right) \sin \phi = 2 \frac{y}{b} \sqrt{1 - \frac{y^2}{b^2}} \cos \phi$$

Squaring on both sides, we get

$$\frac{x^2}{a^2} + \left(1 - \frac{2y^2}{b^2}\right)^2 \sin^2 \phi = \frac{2x}{a} \left(1 - \frac{2y^2}{b^2}\right) \sin \phi$$

$$= \frac{4y^2}{b^2} \left(1 - \frac{y^2}{b^2}\right) \cos^2 \phi$$

$$\frac{x^2}{a^2} + \sin^2 \phi + \frac{4y^4}{b^4} \sin^2 \phi - \frac{4y^2}{b^2} \sin^2 \phi - \frac{2x}{a} \sin \phi + \frac{4xy^2}{ab^2} \sin \phi$$

$$= \frac{4y^2}{b^2} \cos^2 \phi - \frac{4y^4}{b^4} \cos^2 \phi$$

$$\frac{x^2}{a^2} + \sin^2 \phi - \frac{2x}{a} \sin \phi + \frac{4y^4}{b^4} (\sin^2 \phi + \cos^2 \phi) - \frac{4y^2}{b^2} (\sin^2 \phi + \cos^2 \phi)$$

$$+ \frac{4xy^2}{ab^2} \sin \phi = 0$$

$$\frac{x^2}{a^2} + \sin^2 \phi - \frac{2x}{a} \sin \phi + \frac{4y^4}{b^4} - \frac{4y^2}{b^2} + \frac{4xy^2}{ab^2} \sin \phi = 0$$

$$\left(\frac{x}{a} - \sin \phi\right)^2 + \frac{4y^4}{b^4} - \frac{4y^2}{b^2} + \frac{4xy^2}{ab^2} \sin \phi = 0$$

$$\left(\frac{x}{a} - \sin \phi\right)^2 + \frac{4y^2}{b^2} \left(\frac{y^2}{b^2} + \frac{x}{a} \sin \phi - 1\right) = 0$$

This is the eq of a curve having two loops, which is the resultant path.

Here we shall consider the following Spl Cases.



$$\frac{x}{a} = 2 \sin \omega t \cos \omega t \cdot \cos \phi + (1 - 2 \sin^2 \omega t) \sin \phi$$

$\uparrow$   $\cos \omega t$   $\uparrow$   $\sin \omega t$

$$\frac{x}{a} = 2 \frac{y}{b} \sqrt{1 - \frac{y^2}{b^2}} \cos \phi + \left(1 - \frac{2y^2}{b^2}\right) \sin \phi$$

$$\frac{x}{a} - \left(1 - \frac{2y^2}{b^2}\right) \sin \phi = 2 \frac{y}{b} \sqrt{1 - \frac{y^2}{b^2}} \cos \phi$$

Squaring on both sides, we get

$$\begin{aligned} \frac{x^2}{a^2} + \left(1 - \frac{2y^2}{b^2}\right)^2 \sin^2 \phi &= \frac{2x}{a} \left(1 - \frac{2y^2}{b^2}\right) \sin \phi \\ &= \frac{4y^2}{b^2} \left(1 - \frac{y^2}{b^2}\right) \cos^2 \phi. \end{aligned}$$

$$\begin{aligned} \frac{x^2}{a^2} + \sin^2 \phi + \frac{4y^4}{b^4} \sin^2 \phi - \frac{4y^2}{b^2} \sin^2 \phi - \frac{2x}{a} \sin \phi + \frac{4xy^2}{ab^2} \sin \phi \\ = \frac{4y^2}{b^2} \cos^2 \phi - \frac{4y^4}{b^4} \cos^2 \phi \end{aligned}$$

$$\begin{aligned} \frac{x^2}{a^2} + \sin^2 \phi - \frac{2x}{a} \sin \phi + \frac{4y^4}{b^4} (\sin^2 \phi + \cos^2 \phi) - \frac{4y^2}{b^2} (\sin^2 \phi + \cos^2 \phi) \\ + \frac{4xy^2}{ab^2} \sin \phi = 0 \end{aligned}$$

$$\frac{x^2}{a^2} + \sin^2 \phi - \frac{2x}{a} \sin \phi + \frac{4y^4}{b^4} - \frac{4y^2}{b^2} + \frac{4xy^2}{ab^2} \sin \phi = 0$$

$$\left(\frac{x}{a} - \sin \phi\right)^2 + \frac{4y^4}{b^4} - \frac{4y^2}{b^2} + \frac{4xy^2}{ab^2} \sin \phi = 0$$

$$\left(\frac{x}{a} - \sin \phi\right)^2 + \frac{4y^2}{b^2} \left(\frac{y^2}{b^2} + \frac{x}{a} \sin \phi - 1\right) = 0$$

This is the eq of a curve having two loops, which is the resultant path.

Here we shall consider the following Spl Cases.



$$\phi = \pi/4, 0, \pi, 2\pi \text{ ج } \\ \sin \phi = 0 \quad \cos \phi = 1.$$

$$\left(\frac{x}{a}\right)^2 + \frac{4y^2}{b^2} \left(\frac{y^2}{b^2} - 1\right) = 0 \quad - (4)$$

This represent a fig 8 as shown in fig 10(a)

$$\phi = \pi/4 \text{ ج } (ii)$$

In this Case

$$\sin \phi = 1/\sqrt{2}$$

the Eq (3) is

$$\left(\frac{x}{a} - \frac{1}{\sqrt{2}}\right)^2 - \frac{4y^2}{b^2} \left(\frac{y^2}{b^2} - 1 + \frac{x}{a\sqrt{2}}\right) = 0 \quad - (5)$$

this represent a Curve as shown in fig 10(b)



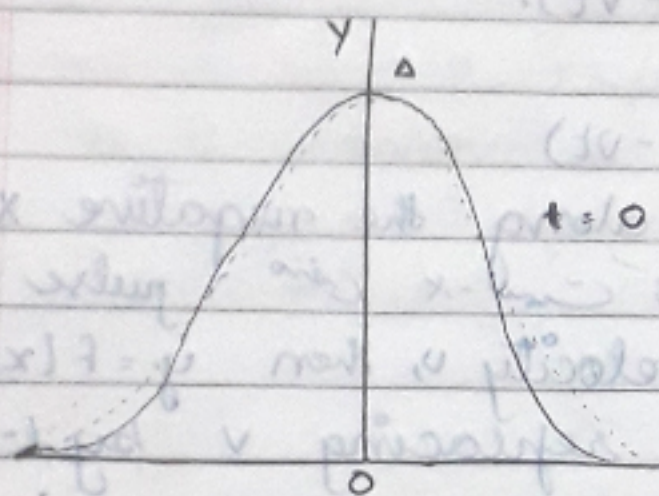
## WAVE MOTION

Gen. wave Eq and its solution:-

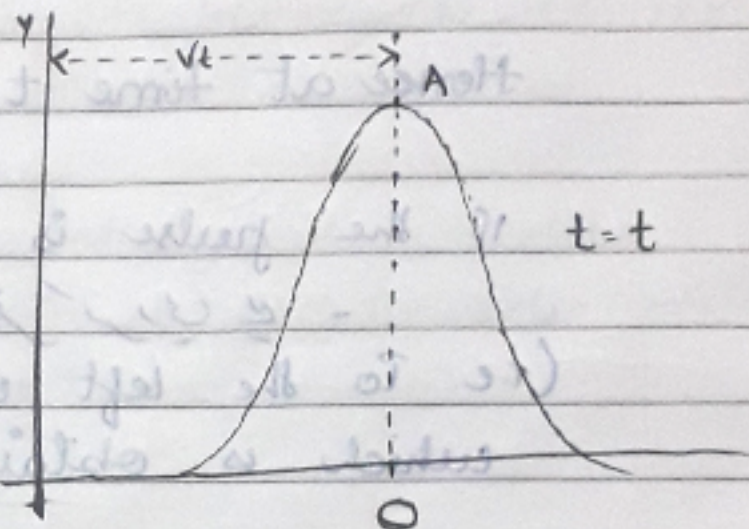
To derive the wave Equation Consider a string stretched in the  $x$ -direction along which a transverse wave is travelling.

Let the shape of the wave (pulse) at  $t=0$  is shown

The pulse is described by  $y=f(x)$  when time  $t=0$



(a) pulse at time  $t=0$



(b) pulse after  $t=t$

Let the pulse is travelling to the right with a velocity  $v$  unchanged (without any deformation).

After a time  $t$ , the pulse reaches a distance  $vt$  along the  $x$ -axis shown in (b)

The wave form now can be represented as  
to which represent the wave form.



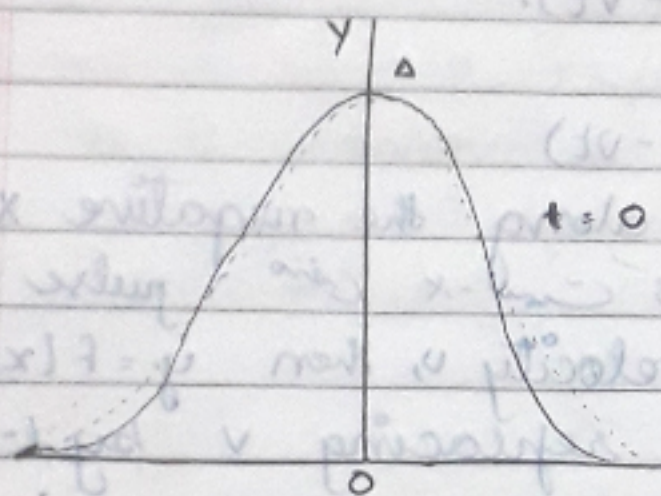
## WAVE MOTION

Gen. wave Eq and its solution:-

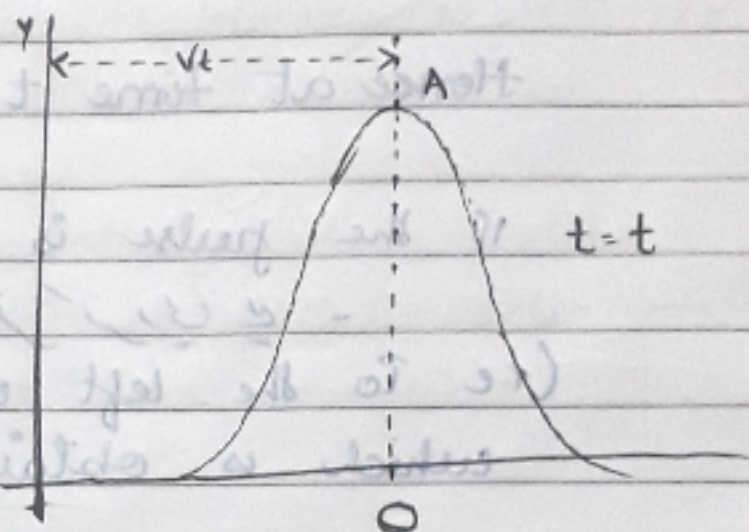
To derive the wave Equation Consider a string stretched in the  $x$ -direction along which a transverse wave is travelling.

Let the shape of the wave (pulse) at  $t=0$  is shown

The pulse is described by  $y=f(x)$  when time  $t=0$



(a) pulse at time  $t=0$



(b) pulse after  $t=t$

Let the pulse is travelling to the right with a velocity  $v$  unchanged (without any deformation).

After a time  $t$ , the pulse reaches a distance  $vt$  along the  $x$ -axis shown in (b)

The wave form now can be represented as  
is  $y=f(x-vt)$  represent the wave form



$$y(x, t) = f(x - vt)$$

Variable  $y$  depend on  $x$  and  $t$  and hence may be written as  $y(x, t)$ . (The above idea may be derived by Galilean Transformation).

Let The pulse time  $t$  is described by  $y = f(x')$

Using Galilean transformation

Galilean transformation

$$x' = (x - vt)$$

Hence at time  $t$ ,

$$y = f(x - vt)$$

If the pulse is travelling along the negative  $x$ -direction (i.e. To the left with a velocity  $v$ , then  $y = f(x + vt)$  which is obtained by replacing  $v$  by  $(-v)$

Hence

$y(x, t) \Rightarrow f(x - vt)$  wave travelling in  $(+)$   $x$ -axis

$y(x, t) \Rightarrow f(x + vt)$  wave travelling in  $(-)$   $x$ -axis

Thus, the wave travelling with a Const. speed  $v$  along  $x$ -axis can be represented by.

ایک  $x$ -محور پر مسئلہ چال سے سفر کر رہی لہر کو اس طرح ظاہر کیا جاسکتا ہے۔

$$y = f(x \pm vt) \quad \text{--- (i)}$$

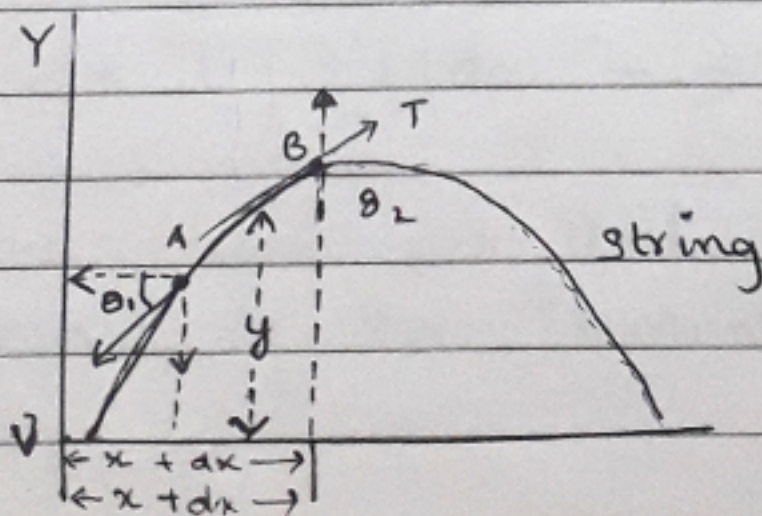


This is due to the fact that tension in the string tends to bring it back to its mean position.

But due to inertia the string overshoot and go to other extreme.

In this way transverse vibrations are setup in the string.

To derive the wave equation for the transverse vibrations of a string fixed b/w two rigid supports and stretched under a tension  $T$  along  $x$ -axis. In displaced position, consider an infinitesimal string element  $AB$  of length  $dx$  between the coordinates  $x$  and  $x+dx$  as shown.



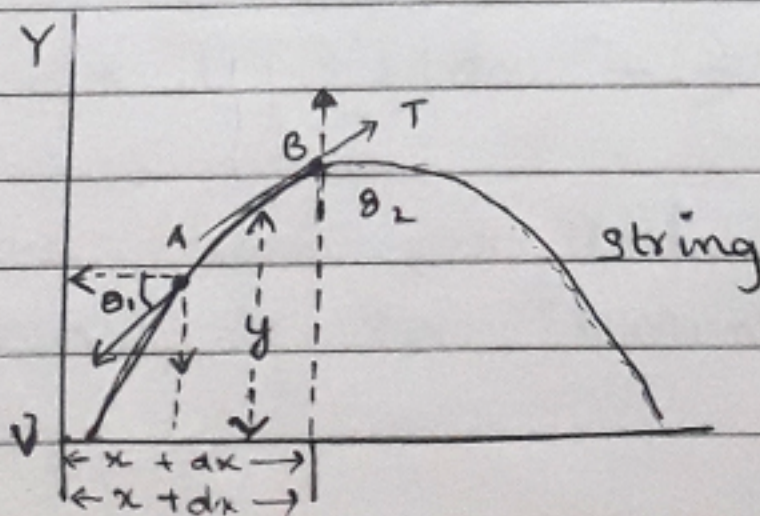


This is due to the fact that tension in the string tends to bring it back to its mean position.

But due to inertia the string overshoot and go to other extreme.

In this way transverse vibrations are setup in the string.

To derive the wave equation for the transverse vibrations of a string fixed b/w two rigid supports and stretched under a tension  $T$  along  $x$ -axis. In displaced position, consider an infinitesimal string element  $AB$  of length  $dx$  between the coordinates  $x$  and  $x+dx$  as shown.



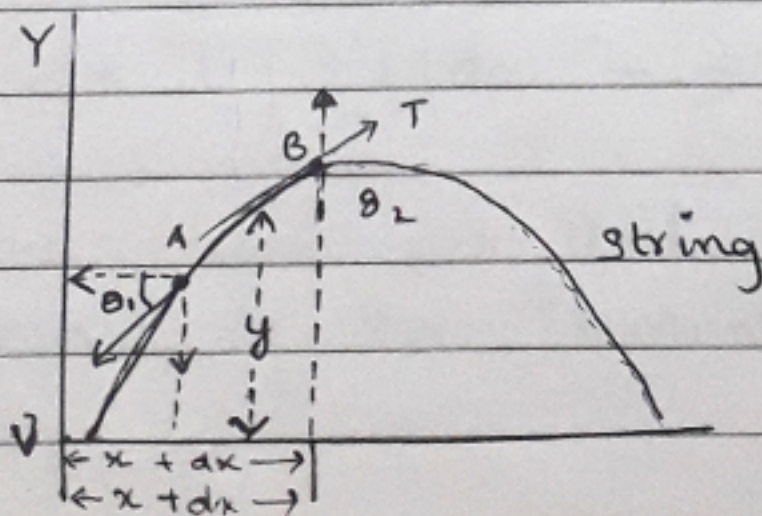


This is due to the fact that tension in the string tends to bring it back to its mean position.

But due to inertia the string overshoot and go to other extreme.

In this way transverse vibrations are setup in the string.

To derive the wave equation for the transverse vibrations of a string fixed b/w two rigid supports and stretched under a tension  $T$  along  $x$ -axis. In displaced position, Consider an infinitesimal string element  $AB$  of length  $dx$  between the coordinates  $x$  and  $x+dx$  as shown.





Taylor's Series Expansion of  $f(x + \Delta x)$   
 $= f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}(\Delta x)^2 + \dots$

$$F_y = T \sin \theta_2 - T \sin \theta_1$$

$$= T [\sin \theta_2 - \sin \theta_1] \quad \text{--- (1)}$$

Since the displacement AB is small  
 Hence  $\theta_1$  and  $\theta_2$  are also small.

Hence

$$\sin \theta_1 \approx \tan \theta_1 \approx \left( \frac{\partial y}{\partial x} \right)_x$$

$$\sin \theta_2 \approx \tan \theta_2 \approx \left( \frac{\partial y}{\partial x} \right)_{x+dx}$$

$$\therefore F_y = T \left[ \left( \frac{\partial y}{\partial x} \right)_{x+dx} - \left( \frac{\partial y}{\partial x} \right)_x \right] \quad \text{--- (2)}$$

Using Taylor series  
 we expand  $\left( \frac{\partial y}{\partial x} \right)_{x+dx}$

$$\left( \frac{\partial y}{\partial x} \right)_{x+dx} = \left( \frac{\partial y}{\partial x} \right)_x + \left( \frac{\partial^2 y}{\partial x^2} \right) dx + \frac{\partial^3 y}{\partial x^3} \frac{(dx)^2}{2!} + \dots$$

Neglecting high power, we have

$$\left( \frac{\partial y}{\partial x} \right)_{x+dx} = \left( \frac{\partial y}{\partial x} \right)_x + \left( \frac{\partial^2 y}{\partial x^2} \right) dx \quad \text{--- (3)}$$

Substituting the value of  $\left( \frac{\partial y}{\partial x} \right)_{x+dx}$  in (2)

$$F_y = T \left[ \left\{ \left( \frac{\partial y}{\partial x} \right)_x + \left( \frac{\partial^2 y}{\partial x^2} \right) dx \right\} - \left( \frac{\partial y}{\partial x} \right)_x \right]$$

$$F_y = T \left( \frac{\partial^2 y}{\partial x^2} \right) dx \quad \text{--- (4)}$$

Let  $m$  be the mass per unit length of the wire  
 then the mass of the element AB will be  $mdx$ .



Taylor's Series Expansion of  $f(x + \Delta x)$   
 $= f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}(\Delta x)^2 + \dots$

$$F_y = T \sin \theta_2 - T \sin \theta_1$$

$$= T [\sin \theta_2 - \sin \theta_1] \quad \text{--- (1)}$$

Since the displacement AB is small  
 Hence  $\theta_1$  and  $\theta_2$  are also small.

Hence

$$\sin \theta_1 \approx \tan \theta_1 \approx \left( \frac{\partial y}{\partial x} \right)_x$$

$$\sin \theta_2 \approx \tan \theta_2 \approx \left( \frac{\partial y}{\partial x} \right)_{x+dx}$$

$$\therefore F_y = T \left[ \left( \frac{\partial y}{\partial x} \right)_{x+dx} - \left( \frac{\partial y}{\partial x} \right)_x \right] \quad \text{--- (2)}$$

Using Taylor series  
 we expand  $\left( \frac{\partial y}{\partial x} \right)_{x+dx}$

$$\left( \frac{\partial y}{\partial x} \right)_{x+dx} = \left( \frac{\partial y}{\partial x} \right)_x + \left( \frac{\partial^2 y}{\partial x^2} \right) dx + \frac{\partial^3 y}{\partial x^3} \frac{(dx)^2}{2!} + \dots$$

Neglecting high power, we have

$$\left( \frac{\partial y}{\partial x} \right)_{x+dx} = \left( \frac{\partial y}{\partial x} \right)_x + \left( \frac{\partial^2 y}{\partial x^2} \right) dx \quad \text{--- (3)}$$

Substituting the value of  $\left( \frac{\partial y}{\partial x} \right)_{x+dx}$  in (2)

$$F_y = T \left[ \left\{ \left( \frac{\partial y}{\partial x} \right)_x + \left( \frac{\partial^2 y}{\partial x^2} \right) dx \right\} - \left( \frac{\partial y}{\partial x} \right)_x \right]$$

$$F_y = T \left( \frac{\partial^2 y}{\partial x^2} \right) dx \quad \text{--- (4)}$$

Let  $m$  be the mass per unit length of the wire  
 then the mass of the element AB will be  $mdx$ .



The force in the upward direction will be

$$F_y = \text{mass} \times \text{acceleration} \\ = (m dx) \left( \frac{\partial^2 y}{\partial t^2} \right) \quad - (5)$$

where  $\partial^2 y / \partial t^2$  is the acceleration on AB.

Comparing Eq (4) and (5),  
we get

$$m \left( \frac{\partial^2 y}{\partial t^2} \right) dx = T \left( \frac{\partial^2 y}{\partial x^2} \right) dx$$

$$\left( \frac{\partial^2 y}{\partial t^2} \right) = \frac{T}{m} \left( \frac{\partial^2 y}{\partial x^2} \right) \quad - (6)$$

The differential Eq of a wave motion is given by

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \quad - (7)$$

From Eq (6) and (7)

$$v = \sqrt{T/m}$$

This gives the velocity of transverse wave along string.



## Reflection at a Boundary: BOUNDARY CONDITION:-

جب  $x=0$  اور  $x=l$  پر  
ہر ایک "ل" تار (string) کی لمبائی ہے۔

What happens when a pulse progressing along stretched string approaches the second end?

Consider a transverse wave travelling along the string.

ہم جانتے ہیں کہ ہر مساوات کا حل

$$y(x,t) = f_1(vt-x) + f_2(vt+x) \quad - (1)$$

جہاں  $f_1(vt-x)$  اور  $f_2(vt+x)$  arbitrary function ہیں

تفعلی  $(vt-x)$  اور  $(vt+x)$  کے

In case of stretched string, some boundary conditions are imposed on two functions and they are no longer arbitrary.

$$x=0; y=0$$

$$0 = f_1(vt) + f_2(vt)$$

$$f_2(vt) = -f_1(vt)$$

This shows that the two functions are same but of opposite sign.

Gen. Expression for net transverse displacement at any point on the string may be written as

$$y = f_1(vt-x) - f_1(vt+x) \quad - (2)$$

$f_1(vt-x) \rightarrow$  travelling to right



$$a_1 + a_2 = 0$$

;

$$b_1 + b_2 = 0$$

$$a_1 = -a_2$$

;

$$b_1 = -b_2$$

Now, Eq 1 become,

$$y = a_1 \sin(\omega t - kx) - a_2 \sin(\omega t + kx)$$

$$+ b_1 \cos(\omega t + kx) + b_2 \cos(\omega t - kx)$$

$$y = a_1 [\sin(\omega t - kx) + \sin(\omega t + kx)]$$

$$+ b_1 [\cos(\omega t + kx) - \cos(\omega t - kx)]$$

we know

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$y = a_1 [\sin \omega t \cos kx - \sin kx \cos \omega t] - [\sin \omega t \cos kx + \cos \omega t \sin kx] + b_1 [\cos \omega t \cos kx - \sin \omega t \sin kx] -$$

$$(\cos \omega t \cos kx + \sin \omega t \sin kx)$$

$$y = -2a_1 \sin kx \cos \omega t + 2b_1 \sin \omega t \sin kx$$

$$y = \sin kx [-2a_1 \cos \omega t + 2b_1 \sin \omega t]$$

thus the 1<sup>st</sup> boundary Condition reduces the opposite travelling waves to stationary waves.

Now, we apply the Boundary Condition -

$$y = 0 \quad x = l \quad \text{at time } t.$$



$$a_1 + a_2 = 0$$

;

$$b_1 + b_2 = 0$$

$$a_1 = -a_2$$

;

$$b_1 = -b_2$$

Now, Eq 1 become,

$$y = a_1 \sin(\omega t - kx) - a_2 \sin(\omega t + kx)$$

$$+ b_1 \cos(\omega t + kx) + b_2 \cos(\omega t - kx)$$

$$y = a_1 [\sin(\omega t - kx) + \sin(\omega t + kx)]$$

$$+ b_1 [\cos(\omega t + kx) - \cos(\omega t - kx)]$$

we know

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$y = a_1 [\sin \omega t \cos kx - \sin kx \cos \omega t] - [\sin \omega t \cos kx + \cos \omega t \sin kx] + b_1 [\cos \omega t \cos kx - \sin \omega t \sin kx] -$$

$$(\cos \omega t \cos kx + \sin \omega t \sin kx)$$

$$y = -2a_1 \sin kx \cos \omega t + 2b_1 \sin \omega t \sin kx$$

$$y = \sin kx [-2a_1 \cos \omega t + 2b_1 \sin \omega t]$$

thus the 1<sup>st</sup> boundary Condition reduces the opposite travelling waves to stationary waves.

Now, we apply the Boundary Condition -

$$y = 0 \quad x = l \quad \text{at time } t.$$



$$\lim_{k \rightarrow 0} k l = 0$$

where

$$k l = n \pi$$

$$n = 1, 2, 3, \dots$$

As  $l$  is const,  $k$  is limited to discrete set values known as Eigenvalues.

$$\frac{2\pi}{\lambda} l = n\pi$$

$$\frac{2l}{\lambda} = n$$

$$\boxed{\lambda = \frac{2l}{n}}$$

$$\lambda f = v \quad ; \quad f = \frac{v}{\lambda}$$

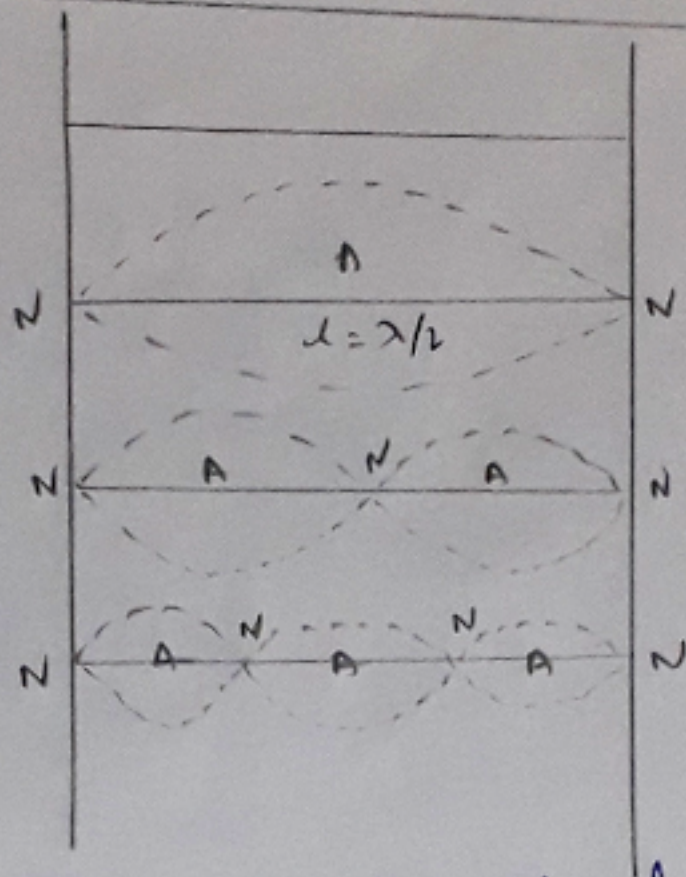
$$f = \frac{nv}{2l}$$

$$f = \frac{n}{2l} \sqrt{\frac{T}{m}}$$

$$v^2 = \frac{T}{m}$$

$$\Rightarrow v = \sqrt{\frac{T}{m}}$$





$$l = N_2$$

Modes of vibration of stretched string clamped at both ends.

The fundamental frequency ( $n=1$ ) is given by

$$f = \frac{1}{2l} \sqrt{\frac{T}{m}}$$

The  $n^{\text{th}}$  harmonic modes of frequency is given by

$$f_n = \frac{n}{2l} \sqrt{\frac{T}{m}}$$